

A q -ANALOGUE OF THE FOUR FUNCTIONS THEOREM

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ABSTRACT. In this article we give a proof of a q -analogue of the celebrated four functions theorem. This analogue was conjectured by Björner and includes as special cases both the four functions theorem and also Björner's q -analogue of the FKG inequality.

1. INTRODUCTION

We denote the set of the first n positive integers by $[n]$ and the power set of $[n]$ by $\mathcal{P}(n)$. Given families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(n)$ we write $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$. Given $f : \mathcal{P}(n) \rightarrow \mathbb{R}$ and $\mathcal{A} \subseteq \mathcal{P}(n)$ we let $f(\mathcal{A}) = \sum_{A \in \mathcal{A}} f(A)$. The four functions theorem of Ahlswede and Daykin [1] states the following.

Theorem 1.1. *Let α, β, γ and δ be functions from $\mathcal{P}(n)$ to the set of non-negative reals satisfying*

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

for every $A, B \in \mathcal{P}(n)$. Then

$$\alpha(\mathcal{A})\beta(\mathcal{B}) \leq \gamma(\mathcal{A} \vee \mathcal{B})\delta(\mathcal{A} \wedge \mathcal{B})$$

for every $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(n)$.

This inequality generalized several well-known inequalities and has found many applications. We refer the interested reader to the books by Anderson [2] and Bollobás [5] and their relevant references for further discussions of this inequality and its applications.

To state the next result we need to recall some facts from lattice theory. We refer the reader to the classical book of Birkhoff [3] for further information on lattices. A *lattice* L is a partially ordered set in which every pair of elements $x, y \in L$ has a unique least upper bound (denoted by $x \vee y$) and a unique greatest lower bound (denoted by $x \wedge y$). The lattice L is called *distributive* if for every $x, y, z \in L$ the following holds:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

It is easy to check that $\mathcal{P}(n)$ is a distributive lattice. Birkhoff's representation theorem asserts that every finite distributive lattice is isomorphic to a sublattice of $\mathcal{P}(n)$ for some n . In particular, Theorem 1.1 has the following immediate consequence.

Corollary 1.2. *Let L be a finite distributive lattice and let α, β, γ and δ be functions from L to the set of non-negative reals satisfying*

$$\alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y)$$

for every $x, y \in L$. Then

$$\alpha(X)\beta(Y) \leq \gamma(X \vee Y)\delta(X \wedge Y)$$

for every $X, Y \subseteq L$.

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Indeed to prove the above corollary we just embed L in $\mathcal{P}(n)$ for some suitable n , extend α, β, γ and δ to be 0 outside L and then apply Theorem 1.1.

Let L be a lattice. A function $f : L \rightarrow \mathbb{R}$ is called *increasing* if $f(x) \leq f(y)$ whenever $x \leq y$ and *decreasing* if $f(x) \leq f(y)$ whenever $x \geq y$. A function μ from L to the set of non-negative reals is said to be *log-supermodular* if it satisfies

$$\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$$

for every $x, y \in L$. It is usually convenient to think of μ as a measure on L . We take this approach here and we thus define

$$\int f d\mu := \sum_{x \in L} f(x)\mu(x).$$

The following correlation inequality due to Fortuin, Kasteleyn and Ginibre [6] is known as the FKG inequality.

Corollary 1.3. *Let L be a finite distributive lattice, μ a log-supermodular function on L and f, g functions from L to the set of non-negative reals which are either both increasing or both decreasing. Then*

$$\int f d\mu \int g d\mu \leq \int 1 d\mu \int f g d\mu.$$

This can be proved by applying Corollary 1.2 with $\alpha = f\mu, \beta = g\mu, \gamma = \mu$ and $\delta = fg\mu$. The FKG inequality originally arose in the study of Ising ferromagnets and the random cluster model. It later found many application in extremal and probabilistic combinatorics. Recently, Björner obtained a q -analogue of this inequality. Before stating it we need to introduce some further notation.

Given a finite lattice L and an element x of L the *rank* or *height* of x is the length of the longest chain having x as a maximal element and is denoted by $r(x)$. If L is distributive then the rank function satisfies the following *modular law*:

$$r(x) + r(y) = r(x \vee y) + r(x \wedge y) \text{ for every } x, y \in L.$$

Given a finite distributive lattice L with rank function r and functions $f, \mu : L \rightarrow \mathbb{R}$ we define the polynomial

$$P_\mu(f; q) = \int f(x)q^{r(x)}d\mu.$$

Finally, given polynomials $P(q), R(q) \in \mathbb{R}[q]$, we write $P(q) \ll R(q)$ to denote that all coefficients of the polynomial $R(q) - P(q)$ are non-negative reals.

Björner's q -analogue of the FKG-inequality [4] reads as follows.

Theorem 1.4. *Let L be a finite distributive lattice, μ a log-supermodular function on L and f, g functions from L to the set of non-negative reals which are either both increasing or both decreasing. Then*

$$P_\mu(f; q)P_\mu(g; q) \ll P_\mu(1; q)P_\mu(fg; q).$$

The FKG inequality is obtained from the above result by putting $q = 1$. Several applications of Theorem 1.4 can be found in [4]. It is natural to ask whether there the corresponding q -analogue of the four functions theorem is true. Answering a question of Björner [4] we prove the following result.

Theorem 1.5. *Let L be a finite distributive lattice and let α, β, γ and δ be functions from L to the set of non-negative reals satisfying*

$$\alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y)$$

for every $x, y \in L$. Then

$$\sum_{x \in X} \alpha(x) q^{r(x)} \sum_{x \in Y} \beta(x) q^{r(x)} \ll \sum_{x \in X \vee Y} \gamma(x) q^{r(x)} \sum_{x \in X \wedge Y} \delta(x) q^{r(x)}.$$

for every $X, Y \subseteq L$.

It is easy to see that Theorem 1.5 includes both Theorem 1.1 and Theorem 1.4 as special cases. We thus obtain a new proof of Theorem 1.4. Unfortunately we do not obtain a new proof of Theorem 1.1 as this theorem itself will be used in the proof of Theorem 1.5.

We give the proof of Theorem 1.5 in the next section. In Section 3 we discuss a stronger conjecture which turns out to be false. In the proof of Theorem 1.5 we will need to use a result which although not a corollary of Birkhoff's representation theorem, it is a simple consequence of its proof. We thus add an appendix with a proof of this result.

2. PROOF OF THEOREM 1.5

We claim that it is enough to prove Theorem 2.1 in the case when L is the Boolean lattice $\mathcal{P}(n)$ and $X = Y = \mathcal{P}(n)$. I.e. it is enough to prove the following theorem.

Theorem 2.1. *Let α, β, γ and δ be functions from $\mathcal{P}(n)$ to the set of non-negative reals satisfying*

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

for every $A, B \in \mathcal{P}(n)$. Then

$$\sum_{A \in \mathcal{P}(n)} \alpha(A) q^{|A|} \sum_{B \in \mathcal{P}(n)} \beta(B) q^{|B|} \ll \sum_{C \in \mathcal{P}(n)} \gamma(C) q^{|C|} \sum_{D \in \mathcal{P}(n)} \delta(D) q^{|D|}.$$

We begin by showing that it is indeed enough to prove the above theorem.

Theorem 2.1 implies Theorem 1.5. Let $\phi : L \rightarrow \mathcal{P}(n)$ be the embedding given by Theorem A.1. For every $A \in \mathcal{P}(n)$ we let $\alpha'(A)$ to be equal to $\alpha(x)$ if $x \in X$ and $\phi(x) = A$. Otherwise we put $\alpha'(A) = 0$. We define β', γ' and δ' analogously. Observe that the functions α', β', γ' and δ' satisfy the conditions of Theorem 2.1. Indeed, $\alpha'(A)\beta'(B)$ is non-zero only when $\phi(x) = A$ and $\phi(y) = B$ for some $x \in X, y \in Y$. But in this case we have $A \cup B = \phi(x) \cup \phi(y) = \phi(x \vee y)$ and thus $\gamma'(A \cup B) = \gamma(x \vee y)$. Similarly we also have $\delta'(A \cap B) = \delta(x \wedge y)$ and thus $\alpha'(A)\beta'(B) \leq \gamma'(A \cup B)\delta'(A \cap B)$ as required. The inequality now follows as $r(a) = |\phi(a)|$ for every $a \in L$. \square

In fact, instead of Theorem 2.1 we will prove the following stronger assertion.

Theorem 2.2. *Let α, β, γ and δ be functions from $\mathcal{P}(n)$ to the set of non-negative reals satisfying*

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

for every $A, B \in \mathcal{P}(n)$. Then

$$\sum_{A \in \mathcal{P}(n)} \alpha(A)\beta(A^c) \leq \sum_{C \in \mathcal{P}(n)} \gamma(C)\delta(C^c).$$

Before proving Theorem 2.2 we show how it implies Theorem 2.1.

Theorem 2.2 implies Theorem 2.1. We need to prove one inequality for each power of q . For the coefficients of q^k we need to prove that

$$\sum_{|A|+|B|=k} \alpha(A)\beta(B) \leq \sum_{|C|+|D|=k} \gamma(C)\delta(D).$$

Since

$$\sum_{|A|+|B|=k} \alpha(A)\beta(B) = \sum_{F,G} \sum_{\substack{|A|+|B|=k \\ A \cap B = F, A \cup B = G}} \alpha(A)\beta(B),$$

it is enough to prove that for each $F, G \in \mathcal{P}(n)$ the following inequality holds:

$$\sum_{\substack{|A|+|B|=k \\ A \cap B = F, A \cup B = G}} \alpha(A)\beta(B) \leq \sum_{\substack{|C|+|D|=k \\ C \cap D = F, C \cup D = G}} \gamma(C)\delta(D).$$

The inequality is vacuously true unless $F \subseteq G$ and $|F| + |G| = 2k$. In this case, we apply Theorem 2.2 on the functions α', β', γ' and δ' from $\mathcal{P}(G \setminus F)$ to \mathbb{R} , where $\alpha'(A) = \alpha(A \cup F)$ and β', γ' and δ' are defined analogously. Since

$$\alpha'(A)\beta'(B) = \alpha(A \cup F)\beta(B \cup F) \leq \gamma(A \cup B \cup F)\delta((A \cap B) \cup F) = \gamma'(A \cup B)\delta'(A \cap B),$$

Theorem 2.2 gives

$$\sum_{A \in \mathcal{P}(G \setminus F)} \alpha'(A)\beta'(A^c) \leq \sum_{C \in \mathcal{P}(G \setminus F)} \gamma'(C)\delta'(C^c).$$

or equivalently

$$\sum_{A \in \mathcal{P}(G \setminus F)} \alpha(A \cup F)\beta(G \setminus A) \leq \sum_{C \in \mathcal{P}(G \setminus F)} \gamma(C \cup F)\delta(G \setminus C).$$

But this is exactly the inequality we wanted to prove. \square

We need one more lemma before proceeding to the proof of Theorem 2.2.

Lemma 2.3. *Let α, β, γ and δ be functions from $\mathcal{P}(n)$ to the set of non-negative reals satisfying*

$$\alpha(A)\beta(B) \leq \gamma(B \setminus A)\delta(A \setminus B)$$

for every $A, B \in \mathcal{P}(n)$. Then

$$\alpha(\mathcal{P}(n))\beta(\mathcal{P}(n)) \leq \gamma(\mathcal{P}(n))\delta(\mathcal{P}(n)).$$

Proof. We just apply Theorem 1.1 to the functions α, β', γ' and δ , where $\beta'(B) := \beta(B^c)$ and $\gamma'(C) := \gamma(C^c)$. The lemma follows directly since for any $A, B \in \mathcal{P}(n)$ we have

$$\alpha(A)\beta'(B) = \alpha(A)\beta(B^c) \leq \gamma(A \cup B^c)\delta(A \cap B^c) = \gamma'(B \setminus A)\delta(A \setminus B). \quad \square$$

We are now ready to prove Theorem 2.2. This will complete the proof of Theorem 1.5.

Proof of Theorem 2.2. Let us define $f : \mathcal{P}(n) \rightarrow \mathbb{R}$ by $f(A) = \alpha(A)\beta(A^c)$ and $g : \mathcal{P}(n) \rightarrow \mathbb{R}$ by $g(A) = \gamma(A^c)\delta(A)$. Then

$$\begin{aligned} f(A)f(B) &= (\alpha(A)\beta(B^c))(\alpha(A^c)\beta(B)) \\ &\leq \gamma(A \cup B^c)\delta(A \cap B^c)\gamma(A^c \cup B)\delta(A^c \cap B) \\ &= g(A \cap B^c)g(A^c \cap B) = g(B \setminus A)g(A \setminus B). \end{aligned}$$

Thus, applying Lemma 2.3 with $\alpha = \beta = f$ and $\gamma = \delta = g$, we obtain that

$$(f(\mathcal{P}(n)))^2 \leq (g(\mathcal{P}(n)))^2.$$

The result now follows as all functions used take only non-negative values. \square

3. A COUNTEREXAMPLE TO A STRONGER CONJECTURE

In our attempt to prove Theorem 1.5 we arrived at the following conjecture.

Conjecture 3.1. *Let α, β, γ and δ be functions from $\mathcal{P}(n)$ to the set of non-negative reals satisfying*

$$\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$$

for every $A, B \in \mathcal{P}(n)$. Then

$$\alpha(A)\beta(B) + \alpha(B)\beta(A) \leq \gamma(A)\delta(B) + \gamma(B)\delta(A).$$

Conjecture 3.1 is easily seen to imply Theorem 2.2 and thus Theorem 2.1. (In fact, it is easy to deduce Theorem 2.1 from Conjecture 3.1 directly without going through Theorem 2.2.) It can be checked that the conjecture is true for $n = 1$ and a simple inductive argument which we omit shows that if the conjecture were true in the case $n = 2$ as well, then it would be true for every positive integer n . Unfortunately, it turns out that the conjecture is false in the case $n = 2$ as can be checked by defining α, β, γ and δ as suggested in the following table.

	α	β	γ	δ
\emptyset	0	1	0	1
$\{1\}$	0	1	0	0
$\{2\}$	1	1	1	1
$\{1, 2\}$	0	0	1	0

There are only three pairs (A, B) for which $\alpha(A)\beta(B)$ is non-zero and in each one of them one can check that the inequality $\alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B)$ holds. However, taking $A = \{1\}$ and $B = \{2\}$ we have $\alpha(A)\beta(B) + \alpha(B)\beta(A) = 1 > 0 = \gamma(A)\delta(B) + \gamma(B)\delta(A)$ thus disproving the conjecture.

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APPENDIX A.

In this appendix we give a proof of Birkhoff's representation theorem. It is usually stated without condition (iii) but this condition, which is needed in the proof of Theorem 1.2, follows easily from the usual proof of the theorem.

Theorem A.1. *Let L be a finite distributive lattice. Then there is a positive integer n and an injective function $\phi : L \rightarrow \mathcal{P}[n]$ satisfying*

- (i) $\phi(a \wedge b) = \phi(a) \cap \phi(b)$ for all $a, b \in L$;
- (ii) $\phi(a \vee b) = \phi(a) \cup \phi(b)$ for all $a, b \in L$;
- (iii) $r(a) = |\phi(a)|$ for all $a \in L$.

Proof. We say that a non-zero element x of L is join-irreducible if $a \vee b < x$ whenever $a < x$ and $b < x$. (Recall that every finite lattice L has a unique minimal element which is called the zero of L .) Let $X = \{x_1, \dots, x_n\}$ be the set of join-irreducible elements of L . By reordering the elements of X , we may assume that if $x_i < x_j$ then $i < j$. For each $a \in L$

define $\phi(a) = \{i : x_i \leq a\}$. To see that ϕ is injective observe that if $a \not\leq b$ then the set $S = \{x \in L : x \leq a, x \not\leq b\}$ is non-empty (as $b \in S$) and contains a join-irreducible element. Indeed if x is a minimal element of S and there are $c, d \in L$ with $c, d < x$ and $c \vee d = x$, then by minimality of x we have that $c, d \notin S$ and so it must be the case that $c, d \leq b$. But then $x = c \vee d \leq b$, a contradiction. Observe that (i) follows immediately from the definition of ϕ . It also follows immediately from the definition that $\phi(a) \cup \phi(b) \subseteq \phi(a \vee b)$. To complete the proof of (ii) observe that if $x \in X$ with $x \leq a \vee b$ but $x \not\leq a$ and $x \not\leq b$, then $x \wedge a < x, x \wedge b < x$ but $(x \wedge a) \vee (x \wedge b) = x \wedge (a \vee b) = x$, contradicting the fact that $x \in X$. It remains to prove (iii). Let $a \in L$ and suppose that $\phi(a) = \{i_1, \dots, i_k\}$, where $i_1 < \dots < i_k$. Since $0 < x_{i_1} < x_{i_1} \vee x_{i_2} < \dots < x_{i_1} \vee \dots \vee x_{i_k} \leq a$, it follows that $|\phi(a)| = k \leq r(a)$. On the other hand, if $0 < y_1 < y_2 < \dots < y_r = a$ is a longest chain having a as a maximal element, then, since ϕ is injective, we have $\emptyset \subsetneq \phi(y_1) \subsetneq \phi(y_2) \subsetneq \dots \subsetneq \phi(y_r)$ showing that $|\phi(a)| \geq r = r(a)$. This completes the proof of (iii) and thus of the theorem. \square

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